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DEMONSTRATION OF THE IMPOSSIBILITY OF RESOLVING ALGEBRAICALLY GENERAL EQUATIONS OF A DEGREE SUPERIOR TO THE FOURTH.

TRANSLATED FROM SERRET'S COURS D'ALGEBRE SUPERIEURE
BY ALEXANDER EVANS, ESQ., ELKTON, MD.

The properties of the roots of an equation, algebraically resolvable, which we have just demonstrated, have place in all cases whether we are concerned about an equation in which the coefficients have determinate values, or whether we consider the coefficients as indeterminate, and in consequence the roots of the equation as being any quantities whatever, having no dependence among themselves.

Placing ourselves at present in the last point of view, we are about to demonstrate that it is impossible to resolve algebraically, general equations of a degree superior to the fourth.

This theorem was demonstrated for the first time by Abel; but I will present here the very remarkable demonstration of Wantzel.

We will behold in this exact reproduction a merited homage to the memory of a geometer whom death has stricken in the vigor of his talent.

Nevertheless I will suppress certain details, useless here, after the developments which I have given concerning the number of values that a function may acquire.

Let
$$f(x) = 0$$
be an equation of the degree m whose coefficients are indeterminate, and designate by $x_1, x_2, \ldots . x_m$ its m roots which we suppose to be algebraically expressible in functions of the coefficients.*

"If the equation $f(x) = 0$ is satisfied by the value x_1 of x, whatever its coefficients may be, we ought to reproduce x_1 identically by substituting in its expression the rational function corresponding to each radical, since the roots of the equation are then entirely arbitrary.

*The inverted commas indicate all that is borrowed literallyfrom the memoir of Wantzel.

"In like manner every relation between the roots ought to be identical, and will not cease to exist if we replace in it these roots, the one by the other in any manner whatever.

"Let us designate by y the first radical which enters into the value of x_1 in following the order of the calculation, and let $y^n = p$; p will depend immediately upon the coefficients of $f(x) = 0$, and will be expressed by a symmetrical function of the roots $F(x_1, x_2, x_3, \ldots .)$; y will be a rational function $\varphi(x_1, x_2, x_3, \ldots .)$ of the same roots.

"As the function φ is not symmetrical, otherwise the nth root of p would be exactly extractable, it ought to change when we permute two roots, x_1, x_2, for example; but the relation

$$\varphi^n = F$$

will always be satisfied. Moreover the function F being invariable in this permutation, the values of φ are among the roots of the equation $y^n = F$ and we have $\quad \varphi(x_2, x_1, x_3, \ldots) = a\varphi(x_1, x_2, x_3, \ldots)$

a being an nth root of unity.

"If we replace on each side x_1, by x_2, and reciprocally, there results

$$\varphi(x_1, x_2, x_3, \ldots .) = a\varphi(x_2, x_1, x_3, \ldots .),$$

whence by multiplying in order, $a^2 = 1$.

"This result proves that the number n, supposed to be prime, is necessarily equal to 2; *therefore the first radical which presents itself in the value of the unknown ought to be of the second degree.* This is what happens in effect for the equations which we know how to resolve."

The function φ having only two values changes by any transposition, and will not be changed (see the 19th lesson) by a circular permutation of three or five letters, for these permutations are equivalent to an even number of transpositions.

Let us continue the series of operations indicated in order to form the value x_1 of x.

"We will combine the first radical with the coefficients of $f(x) = 0$, or the function φ with symmetrical functions of the roots, by the aid of the first operations of algebra, and in this way we will obtain a function of the roots susceptible of two values and in consequence invariable by circular permutations of three letters.

"The subsequent radicals will give besides functions of the same kind, if they be of the second degree. Let us suppose that we have reached a radical for which the equivalent rational function may not be invariable by these permutations. Let us always designate it by

$$y = \varphi(x_1, x_2, x_3, \ldots .);$$

in the equation $y^n = p$ we make again

$$p = F(x_1, x_2, x_3, \ldots .);$$

this function will be no longer symmetrical, but only invariable by circular permutations of three letters. If we replace x_1, x_2, x_3, by x_2, x_3, x_1, in φ, the relation $\varphi^n = F$ will always subsist; and since F does not change by this substitution, it becomes

$$\varphi(x_2, x_3, x_1, x_4, \ldots .) = a\varphi(x_1, x_2, x_3, x_4, \ldots .),$$

a designating an nth root of unity."

In making in this equation the circular substitution

$$\begin{pmatrix} x_1, & x_2, & x_3, \\ x_2, & x_3, & x_1, \end{pmatrix},$$

and repeating this substitution a second time we will have

$$\varphi(x_3, x_1, x_2, x_4, \ldots .) = a\varphi(x_2, x_3, x_1, x_4, \ldots .),$$
$$\varphi(x_1, x_2, x_3, x_4, \ldots .) = a\varphi(x_3, x_1, x_2, x_4, \ldots .),$$

and by multiplying the three preceding equations "we conclude that

$$a^3 = 1$$

and so n will be equal to 3.

"If the number of the quantities x_1, x_2, x_3, x_4, is greater than four, or if the degree of the equation $f(x) = 0$ is higher than the fourth, we will be able to perform in φ a circular substitution of five letters in replacing

$$x_1, x_2, x_3, x_4, x_5,$$

by

$$x_2, x_3, x_4, x_5, x_1;$$

the function F will not change, and we will have

$$\varphi(x_2, x_3, x_4, x_5, x_1, \ldots .) = a\varphi(x_1, x_2, x_3, x_4, x_5, \ldots .),$$

then in repeating the same substitution from all quarters

$$\varphi(x_3, x_4, x_5, x_1, x_2, \ldots .) = a\varphi(x_2, x_3, x_4, x_5, x_1, \ldots .),$$

. . . .

. . . .

. . . .

By multiplication we obtain

$$x^5 = 1 \qquad \text{which carries with it}$$
$$a = 1$$

since a is a cube root of unity. So the function φ is invariable for the circular permutations of five letters."

Therefore in consonance with a theorem demonstrated in the nineteenth lesson, the function φ is also invariable for the circular permutations of three letters.

"*Thus all the radicals included in the root of a general equation of a degree superior to the fourth, should be equal to the rational functions of the invariable roots* by *the circular permutations of three roots.* In substituting these functions in the expression of x_1 we come to an equality of the form

$$x_1 = \psi(x_1, x_2, x_3, x_4, x_5, \ldots),$$

which ought to be identical; but which is impossible since the second member remains invariable when we replace x_1, x_2, x_3, by x_2, x_3, x_1, whilst the first evidently changes.

"Therefore it is impossible to resolve by radicals a general equation of the fifth degree, or of a higher degree.

"The precedent demonstration shows us at the same time that for the equations of the third and of the fourth degree, the first radical in the order of the operations ought to be a quadratic radical, and the second a cubic radical. These circumstances present themselves in effect in the formulas given by Lagrange and other geometers."

NOTE —In order to well understand the developments upon which we are about to enter, it is necessary to form for ourself a precise idea of the operation which we have designated by the word *substitution.*

Let $F(a, b, c, \ldots k, l)$ be a function of n letters. If among these n letters we take p at random, $a, b, c, \ldots g$ for example, and after having ranged them in a circle, we put each one of them in place of that which precedes, we say that we have made these p letters suffer a circular permutation, and the substitution

$$\begin{pmatrix} a, b, c, \ldots g \\ b, c, \ldots g, a \end{pmatrix}$$

is called a circular substitution of the order P. That being established we have the following theorem.

Every substitution if it be not circular, is equivalent to several circular substitutions simultaneously effected upon the different letters.

Let us suppose in effect that we shonld impose upon the letters $a, b, c, \ldots f, g$, any substitution whatever; by this substitution a finds itself replaced by a certain letter c for example, c itself will be replaced by a 3rd letter e, and continuing in this manner we shall fall necessarily upon a letter which will find itself replaced by a. But it is evident that the letters we have thus encountered have undergone a circular permutation.

By taking one of the remaining letters and operating in the same manner we shall form a new group of letters which will equally have submitted to a circular permutation, and so on until all the letters are exhausted.

The reasoning of which we are about to make use gives the means of forming immediately the circular substitutions which are equivalent to a given substitution. Let us consider for example the substitution

$$\begin{pmatrix} a, & b, & c, & d, & e, & f, & g, & h, & i, & j, & o \\ h, & o, & d, & f, & b, & j, & a, & g, & e, & c, & i \end{pmatrix}$$

We shall find that it is equivalent to the three following circular substitutions

$$\begin{pmatrix} a, & h, & g \\ h, & g, & a \end{pmatrix} \quad \begin{pmatrix} b, & o, & i, & e \\ o, & i, & e, & b \end{pmatrix} \quad \begin{pmatrix} c, & d, & f, & j \\ d, & f, & j, & c \end{pmatrix}$$

The same proceeding ought also to be employed when we wish to ascertain whether a substitution be circular or not. So we will find that the substitution

$$\begin{pmatrix} a, & b, & c, & d, & e, & f, & g, & h, & i, & j, & o \\ g, & d, & f, & j, & a, & o, & c, & i, & b, & e, & h \end{pmatrix}$$

is circular, for we may write it in the manner following

$$\begin{pmatrix} a, & g, & c, & f, & o, & h, & i, & b, & d, & j, & e \\ g, & c, & f, & o, & h, & i, & b, & d, & j, & e, & a \end{pmatrix}$$

If after having effected a circular substitution upon p letters we repeat $1, 2, 3, \ldots p - 1$ times the same substitution, we shall obtain p different arrangements, but in making this substitution once more we shall reproduce the primitive arrangement.

We designate by the word *transposition* the circular substitution of two letters, that is to say the operation which consists in simply exchanging these two letters the one for the other, and we indicate by the abridged notation (a, b) the transposition of the letters a and b.

It is evident that every substitution, circular or not, is equivalent to a series of transpositions.

For suppose it concerns us to operate any substitution upon the letters

$$a, b, c, \ldots f, g$$

we will cause a to take the new place which it ought to occupy, by a transposition; this being done another transposition will lead b to the place which it ought to occupy, and so on, until all the letters have taken the places which we wish to assign to them.

The following partial list of references is added for the benefit of students who may wish to consult the original authorities bearing on this subject.

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[As it was thought that the foregoing demonstration, in itself would be found rather obscure and unsatisfactory, we suggested to Mr. Evans that he supply, in the form of a note to the demonstration, for the benefit of such readers as have not access to Serret's book, a translation of so much of the 19th lesson referred to as specially relates to the demonstration under consideration. To that suggestion Mr. Evans has responded with a somewhat extended note.

As our space at present will not permit the introduction of Mr. Evans' note in full, we have inserted above that part of it which relates to *circular substitution* and *transpositson*, together with the references given by Mr. Evans for the benefit of students who may wish to consult the original discussions bearing upon the subject, and will, in future Nos. of the ANALYST, insert any queries that may present themselves to students of this demonstration, and such discussions as the queries may elicit from correspondents who have critically examined the demonstration.—Ed.]